S3 DMS MODULE : 2

FUNDAMENTALS OF COUNTING THEORY

Syllabus

* The Rule of sum – Extension of Sum Rule.
* The Rule of Product - Extension of Product Rule.
* Permutations.
* Combinations
* The Binomial Theorem (without proof)
* Combination with Repetition.
* The Pigeon hole Principle.
* The Principle of Inclusion & Exclusion Theorem (no proof)
* Generalization of the Principle
* Derangements.

Introduction

Enumeration or counting is a process that a student learns when first studying arithmetic. This chapter provides some warning about the seriousness and difficulty of mere counting. Enumeration has applications in areas such as

* coding theory
* probability & statistics
* analysis of algorithms etc.

Basic Principles of counting

I. The Rule of sum:

If a first task can be performed in 'm' ways, while a second task can be performed in 'n' ways and the two tasks cannot be performed simultaneously then performing either task can be accomplished in any one of $m+n$ ways.

Note

The occurrence of a task say 'm' ways are assumed to be distinct, unless there will be a contradiction. This will be true throughout the entire text.

Extension of sum Rule

If tasks $T_1, T_2, \ldots T_m$ can be done in $n_1, n_2, \ldots n_m$ ways respectively and no two of these tasks can be performed at the same time, then the number of ways to do one of these tasks is $n_1+n_2+\ldots+n_m$.

Eg:- 1) A college library has 40 text books on sociology and 50 textbooks dealing with anthropology. In how many ways a student at this college can select text books inorder to learn more about one or the other of these two subjects.

Task: Selection of one Text

ways: $\boxed{50 \text{ Sociology}}$ $\boxed{40 \text{ Anthropology}}$

Total = 50+40=90 ways

Ans:- 50 texts on sociology.
40 textbooks on anthropology.
∴ A student can ~~to~~ select among 90 textbooks by the rule of sum. $(50+40=90)$

2) If a student can choose a project either 20 from mathematics or 35 from CS or 15 from engineering. In how many ways a student can choose a project.

Task: Choose a Project

$\boxed{20 \text{ Maths}}$ $\boxed{35 \text{ CS}}$ $\boxed{15 \text{ Engg}}$.

Ans:- $20+35+15 = 70$ ways a student can choose a project, by extension of sum Rule.

3) A university representative is to be chosen
H.W either from 200 teaching or 300 non teaching employees. In how many ways a representative can be chosen.

4) A computer Science instructor has 7 different
H.W introductory books each on C++, JAVA and PERL. ~~to learn a first programming language.~~ In how many ways a student who is interested in learning any one of these languages can select a text book.

II The Rule of Product:

If a procedure can be broken down into first and second stages and if there are 'm' possible outcomes for the first stage and if, for each of these outcomes, there are 'n' possible outcome for the second stage, then the total procedure can be carried out, in the designated order, in mn ways.

Extension of product Rule:

Suppose a procedure consists of performing tasks $T_1, T_2, \ldots T_m$ in that order. Suppose that task T_i can be performed in n_i ways after the Tasks $T_1, T_2, \ldots T_{i-1}$ are performed, then the number of ways the procedure can be executed in the designated order is $n_1 n_2 n_3 \cdots n_m$

1) A drama club of Central university is holding tryouts for a spring play. With six men and eight women auditing for the leading male and female roles; In howmany ways the director can cast his leading couple.

Ans:- procedure:- Select a leading Couple.

↓

Stage: 1 $\boxed{\text{Men}}$ → 6 persons

↓

Stage: 2 $\boxed{\text{Women}}$ → 8 womens

∴ By the rule of Product, the number of ways the director can select a leading couple is $6 \times 8 = 48$ ways.

2) Consider the manufacturing process of license plates consisting of 2 letters followed by four digits, if no letter or digit can be repeated. In how many ways it is possible.

Ans:- Procedure: Manufacture of License Plate

Stage: 1 $\boxed{\text{Select 1st letter}}$ → 26 letters (26 English Alphabets)

Stage: 2 $\boxed{\text{Select 2nd letter}}$ → 25 letters (because one letter is already used)

Stage: 3 $\boxed{\text{Select 1st digit}}$ → 10 numbers (0, 1, 2, 3, 4, 5, 6, 7, 8, 9)

Stage: 4 $\boxed{\text{Select 2nd digit}}$ → 9 numbers

Stage: 5 $\boxed{\text{Select 3rd digit}}$ → 8 "

Stage: 6 $\boxed{\text{Select 4th digit}}$ → 7 numbers.

Ans:- No of ways $= 26 \times 25 \times 10 \times 9 \times 8 \times 7 = 3,276,000$

3) For the above problem(2) if repetitions of letters and digits are allowed; In how many ways the license plates can be manufactured.

Ans:- $\underline{\underline{26 \times 26 \times 10 \times 10 \times 10 \times 10}}$

4) If repetitions are allowed and the plates have only vowels (A, E, I, O, U) and even digits. (Here consider, zero as an even integer.)

0, 2, 4, 6, 8 → 5 even digits ~~integers~~.

Ans:- $\underline{\underline{5 \times 5 \times 5 \times 5 \times 5 \times 5}}$

5) A tourist can travel form Hyderabad to Tirupati in 4 ways (by Plane, Train, bus or Taxi)
H.W
He can then travel from Tirupati to Tirumala hills in 5 ways (RTC bus, taxi, ropeway, motorcycle or walk). In how many ways a tourist can travel from Hyderabad to Tirumala hills.

6) "LP" brand shirt available in 12 colours, has a male & female version. It comes in 4 sizes
H.W
for each sex, comes in 3 makes of economy, standard and luxury. Then how many ways different types of shirts produced?

7) Application of both Product & sum Rule.

A hotel offers 12 kinds of sweets, 10 kinds of hot tiffins and 5 kinds of beverages (hot tea, hot coffee, juice, coke, ice cream). The breakfast consists of a sweet and a hot beverage or a hot tiffin and a cold beverage. In how many ways the breakfast can be ordered.

Task : $\boxed{\text{ORDER OF BREAKFAST}}$

Sweet & Hot beverage	Hot tiffin & cold beverage
Stage: 1 Sweet 12 kinds	Stage 1 Hot tiffins 10 kinds
Stage: 2 Hot beverge 2 kinds	Stage: 2 Cold Beverages 3 kinds
12×2 ways	10×3 ways
24 ways (By product Rule)	30 ways (By product Rule)

OR.

$\therefore$ Total ways of ordering a breakfast is

$24 + 30 = 54$ ways. By Sum Rule

Answers. (Pg. No: 3)

3 Task: 1 University Representative

ways: 200 teaching | 300 non teaching..

$\therefore$ $\left.\begin{array}{r}200+300\\=500\end{array}\right\}$ possible ways are there to choose a representative.
By Sum Rule.

4. Task: Choose a text book. (Pge. 3)

ways:- $\boxed{\text{C++ 7 books}}$ $\boxed{\text{JAVA 7 books}}$ $\boxed{\text{PERL 7 books}}$.

$\therefore$ Total ways = $7+7+7=21$ by Extn Rule of Sum.

Rule of Product Answers (P.No, 6, 7)

5. Procedure: Hydrabad — Tirumala travel.

Stage 1: $\boxed{\text{Hyderabad — Tirupati}}$ 4 ways.

Stage 2:- $\boxed{\text{Tirupati — Tirumala}}$ 5 ways

$\therefore$ By product Rule, total number of ways $= 4\times5 = 20$

6) Procedure: Production of shirt

Statge: 1 $\boxed{\text{choose a colour}}$ 12 (colours)

Stage: 2 $\boxed{\text{select a gender}}$ 2 (male & female)

Stage: 3 $\boxed{\text{Select a size}}$ 4 (sizes)

Stage: 4 $\boxed{\text{select a quality}}$ 3 types

$\therefore$ By Rule of product $12 \times 2 \times 4 \times 3 =$ no of ways.

$= \underline{288}$

More Problems

1. During a local campaign, eight Republican & five Democratic candidates are nominated for the president of the school board.

(a) If the president is to be one of these candidate, how many possibilities are there for the eventual winner. Which counting principle is used here.

(b) How many possibilities exist for a pair of candidates (one from each party) to oppose each other for the eventual election, & which is the counting principle used.

Ans:- (a)

President of the school board

$\boxed{8 \text{ Republican}}$ $\quad$ $\boxed{5 \text{ Democratic}}$

$\therefore$ Total number of possibilities for the winner $= 8+5 = \underline{\underline{13}}$ by sum Rule

(b) procedure: President

Stage 1 $\boxed{8 \text{ Rep}}$

Stage 2 $\boxed{5 \text{ Demo}}$

Total number of possibilities for a pair of candidates one from each part $= 8 \times 5 = \underline{\underline{40}}$

(By product Rule)

2) Buick automobiles come in 4 models, 12 colours, three engine sizes, and two transmission types.
(a) How many distinct Buicks can be manufactured?
(b) If one of the available colour is blue, how many different blue Buicks can be manufactured.

Ans:- a) Buicks

Stage: 1 Models - 4
Stage: 2 Colour - 12
Stage: 3 Engine Size - 3
Stage: 4 Transmission type - 2

∴ Total numbers of Buicks
$= 4 \times 12 \times 3 \times 2 = \underline{\underline{288}}$

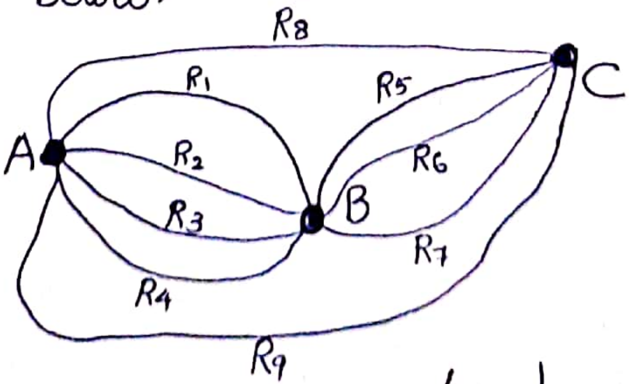
(Rule of product)

b) Blue Buicks

Models - 4
Colour - Blue (1)
Engine size - 3
Transm type - 2

Total number of blue Buicks $= 4 \times 3 \times 2$
$= \underline{\underline{24}}$

3) Three small towns, designated by A, B and C are interconnected by a system of two way road, as shown below.



a) In how many ways can Linda from town A to town C.

b) How many ways can Linda travel from town A to town C and back to town A

Ans:- a) Travel from A to C.

OR   OR

$\boxed{R_8} \rightarrow$ 1 way $\boxed{R_9}$ 1 way $\boxed{A \rightarrow B}$ 4 ways

$\boxed{B \rightarrow C}$ 3 ways

$4 \times 3 = 12$ ways (product Rule)

$\therefore$ By Rule of sum $1+1+12 = \underline{\underline{14 \text{ ways}}}$.

b) $A \longleftrightarrow C$ (To and frow journey)

From the previous problem we know that there are 14 ways from $A \rightarrow C$

|||rly 14 ways from $C \longrightarrow A$

procedure: $\underline{A \rightarrow C \longrightarrow A}$

Stage 1: $\boxed{A \rightarrow C}$ 14 ways

Stage 2: $\boxed{C \rightarrow A}$ 14 ways

$\therefore$ Total ways from A to C & C to A $= \underline{\underline{14 \times 14}}$

4) Patter's Pastry Parlor offers eight different kinds of pastry and six different kinds of muffins. In addition to bakery items one can purchase small, medium or large containers of the following beverages.

coffee (black, with cream, with sugar, or with cream & sugar)

tea (plain, with cream, with sugar, or with cream & sugar, with lemon, or with lemon & sugar)

hot cocoa

and orange juice. When Carol comes to Patter's, in how many ways can she order

a) One bakery item and one medium sized beverage for herself?

b) One bakery item and one container of coffee for herself and one muffin and one container of tea for her Boss Ms. Didio?

c) One piece of pastry and one container of tea for herself, one muffin and a container of orange juice for Ms. Didio, and one bakery item and one container of coffee for each of her two assistants, Mr. Talbot and Mrs. Gillis?

Ans:- Pastry − 8
Muffins − 6
Container − 3

Beverages − 4 { coffee 4 types, tea 6 types, hot cocoa 1, orange juice 1 }

5. While on a Saturday shopping spree Jennifer and Tiffany witnessed two men driving away from the front of a jewelry shop, just before a burglar alarm started to sound. Although everything happened rather quickly, when the two young ladies were questioned they were able to give the police the following information about the license plate (which consisted of two letters followed by four digits) on the get-away car. Tiffany was sure that the second letter on the plate was either an O or a Q and the last digit was either a 3 or an 8. Jennifer told the investigator that the first letter on the plate was either a C or a G and that the first digit was definitely a 7. How many different license plates will the police have to check out?

Permutations (Definition)

Linear arrangement of ~~numbers~~ objects are called permutation, when the objects are distinct.

Eg:- In a class of 10 students, five are to be chosen and seated in a row for a picture. How many such linear arrangements are possible.

Ans:- Consider the possition and possible number of students we can choose to fill each position. Filling of each position is a stage of our procedure.

1st position	Second positi	3rd position	4th position	5th position
10	9	8	7	6

By the Rule of product, the total number of arrangements of five students from a group of 10 $= 10 \times 9 \times 8 \times 7 \times 6$ (because repetitions are not allowed.)

$= \underline{30,240}$

Definition

For an integer $n \geq 0$ n factorial ($n!$) is defined by $\boxed{n! = n(n-1)(n-2)\cdots(3)(2)(1)}$

$\boxed{0! = 1}$ (Defined)

The above answer can be expressed the following compact form

$$10\times9\times8\times7\times6 = 10\times9\times8\times7\times6\times\frac{5\times4\times3\times2\times1}{5\times4\times3\times2\times1} = \frac{10!}{5!}$$

ie,

If there are 'n' distinct objects and 'r' is an integer with $1 \leq r \leq n$, then by the rule of product, the number of permutations of ~~r~~ size 'r' for the 'n' objects are

$$P(n,r) = \underset{1^{st}\text{ posi}}{n}\ \underset{2^{nd}\text{ position}}{(n-1)}\ \underset{3^{rd}\text{ position}}{(n-2)}\cdots\underset{r^{th}\text{ position}}{(n-r+1)}$$

$$= n(n-1)(n-2)\cdots n-(r+1)\times\frac{(n-r)(n-r-1)\cdots(3)(2)\cdot1}{(n-r)(n-r-1)\cdots(3)(2)(1)}$$

$$\boxed{P(n,r) = \frac{n!}{(n-r)!}}$$ If repetition are not allowed.

Note :- If repitions are allowed, the number of permutations of n objects, taken r at a time $= n^r$ (by product rule)

1) What is the number of words of three distinct letters formed from the letter of the word "JNTU"?

Ans:- $P(4,3) = \frac{4!}{(4-3)!} = 4! = 4 \times 3 \times 2 \times 1 = \underline{24}$ (distinct means not repeated)

2) What is the number of possible six letter word from the letters of the word "JNTU".

Ans:- $4^6 = 4096$ (six letter word from 4 letter word means repeatitions are possible)

$\therefore \underline{n^r}$

3) Consider the word "COMPUTER"

(a) What is the number of permutations of the word "COMPUTER"

(b) What is the number of permutation of the word, if 5 of them are used.

(c) What is the number of possible 12 letter sequence using COMPUTER.

ⓐ $8! = P(8,8) = \frac{8!}{(8-8)!} = \frac{8!}{0!} = 8!$

ⓑ $P(8,5) = \frac{8!}{(8-5)!} = \frac{8!}{3!} = \frac{1\times2\times3\times4\times5\times6\times7\times8}{1\times2\times3} = 6720$

ⓒ 12 letter sequence from an 8 letter word means repititions are possible

$\therefore 8^{12} = \underline{\underline{6.872\times10^{10}}}$

4) In how many ways can eight men & 8 women be seated in a row if

ⓐ any person may sit next to any other.

ⓑ men & women must occupy alternate seats.

ⓒ Generalise the result for 'm' men & 'm' women

Ans:- ⓐ 8 Men + 8 Women = 16 persons.

$\therefore$ Number of permutations of ~~8~~ 16 chosen from 16 objects $= P(16,16) = \underline{\underline{16!}}$

ⓑ

M	W	M	W	M	W	M	W	M	W	M	W	M	W	M	W
8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1

Man sitting first : The number of permutation $= 8!\,8!$
$= (8!)^2$

W	M	W	M	W	M	W	M	W	M	W	M	W	M	W	M
8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1

Women sitting first : The number of permutations
$= 8!\,8! = (8!)^2$

$\therefore$ Total number of ways men & women occupy alternatively is $\underline{\underline{2(8!)^2}}$

(c) Generalisation

Any person may sit : $(2m)!$

Men & Women sit alternatively: $2(m!)^2$

5) List all the permutations for the letters a, c, t

Ans:-

6) (a) How many permutations are there for the eight letters a, c, f, g, i, t, w, x ?

(b) Consider the permutation in part (a). How many start with the letter t ?

(c) How many starts with letter t and ends with letter c.

Ans:-

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Pg no: 19

Ans 5) 3!

6) a) 8!

b) 7!

c) 6!

6 b)

t								
1	7	6	5	4	3	2	1	= 7!

c)

t							c	
1	6	5	4	3	2	1	1	= 6!

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7) The number of arrangements (linear) of the four letters in BALL ?

B A L L, B L A L, B L L A
A B L L, L B L A, L B A L
A L B L, L A B L, L L B A
A L L B, L A L B, L L A B

∴ 12 different arrangements are possible

(It is not $4! = 4 \times 3 \times 2 \times 1 = 24$)

∴ The number of arrangements of the 4 letters in BALL is $\frac{4!}{2} = 12$

8) What is the number of arrangements of all nine letters in DATABASES

Ans:- $\frac{9!}{3!\,2!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times \cancel{4}^{2} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{1 \cdot 2 \cdot 3} \times \cancel{2} \cdot 1}$

$= 30,240$

D → 1, A → 3, T → 1, B → 1, S → 2, E → 1

9) How many seven letter words can be formed using the letters of the word BENZENE ?

BENZENE → 7 total
E → 3 times
N → 2 times.

Ans:- $P(7;3,2) = \frac{7!}{3!\,2!} = 420$

RESULT

The number of permutations of a set of object some of which are alike, ie, The number of permutations of n objects of which n_1 are alike, n_2 are alike, ... n_r are alike is

$$P(n; n_1, n_2 \dots n_r) = \frac{n!}{n_1!\, n_2! \dots n_r!}$$

9) The MASSASAUGA is a brown & white venomous snake indigenous to North America. What is the number of arrangements of all letters in MASSASAUGA

10 letters → total.
A → 4 times
S → 3 times.

Ans:- $P(10; 4, 3) = \frac{10!}{4!\, 3!} = \underline{\underline{25,200}}$

10) In the above question what are the possible arrangements of letters in MASSASAUGA if all four A's are together.

MASSASAUGA

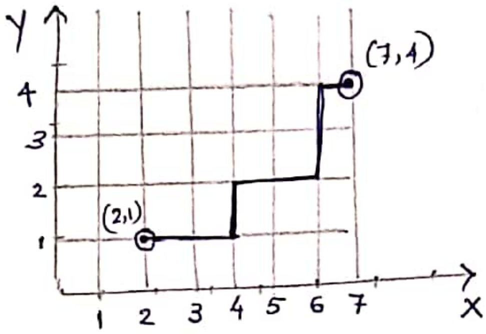
M(AAAA)SSSUG → Total letters = 7
S → 3 times
M, A, U, G → 1 times

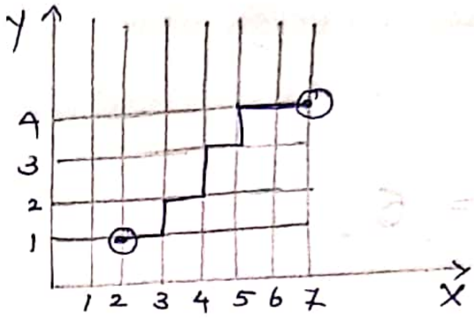
$\therefore P(7; 3) = \frac{7!}{3!}$

$= \underline{\underline{840}}$

11a) Determine the number of paths in the xy plane from (2,1) to (7,4) where each such path is made up of individual steps going one unit to the right (R) or one unit upward (U)

Ans:-





one path is RRURRUUR

another path is RURURURR

in both cases the path is made up of 5 R's and 3 U's. In general the overall path from (2,1) to (7,4) requires $7-2=5$ horizontal moves (x co-ordinate) to right and $4-1=3$ vertical moves (y co-ordinate) upward.

∴ the number of arrangements of 5 R's & 3 U's in RRURRUUR.

$$P(8;5,3) = \frac{8!}{5!\,3!} = \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8}{1\cdot2\cdot3\cdot4\cdot5\times3\cdot2\cdot1} = 56$$ paths are possible.

12 How many distinct paths are there from
a) $(-1,2,0)$ to $(1,3,7)$ in Euclidean three space if each move is one of the following types.

(H): $(x,y,z) \longrightarrow (x+1,y,z)$
(V): $(x,y,z) \longrightarrow (x,y+1,z)$
(A): $(x,y,z) \longrightarrow (x,y,z+1)$

b) How many such paths are there from $(1,0,5)$ to $(8,1,7)$

c) Generalize the results in parts (a) & (b)

13) How many different paths in the XY plane are there from $(0,0)$ to $(7,7)$ if a path proceeds one step at a time by going either one space to the right or one space upward. How many such paths are there from $(2,7)$ to $(9,14)$

14) Find the values of n in each of the following

(a) $P(n,2) = 90$

b) $P(n,3) = 3 \times P(n,2)$

(c) $2\,P(n,2) + 50 = P(2n,2)$

15) Over the internet, data are transmitted in structured blocks of bits called datagrams.

a) In how many ways can the letters in DATAGRAM be arranged?

b) For the arrangements of part (a), how many have all three A's together.

16) (a) How many arrangements are there of all the letters in SOCIOLOGICAL ?

(b) In how many of the arrangements in part (a) are A & G adjacent ?

(c) How many ~~po~~ of the arrangements in part (a) are all the vowels adjacent.

Answers (P. 23 — P. 25)

12 (a) $(-1, 2, 0)$ to $(1, 3, 7)$

$1-(-1) = 2$ x co-ordinate moves

$3-2 = 1$ y co-ordinate moves

$7-0 = 7$ z co-ordinate moves.

$\therefore$ Number of distinct paths from $(-1, 2, 0)$ to $(1, 3, 7)$

$$= \frac{10!}{2!\,1!\,7!} = \underline{\underline{360}}$$

(b) Number of paths from $(1, 0, 5)$ to $(8, 1, 7)$

$8-1 = 7$ x co-ordinate moves

$1-0 = 1$ y co-ordinate move

$7-5 = 2$ z co-ordinate move.

$\therefore$ Number of paths $= \frac{10!}{7! \times 2!} = \underline{\underline{360}}$

Generalization.

Let x, y, z be any integer and let m, n, p be +ve integers. The number of paths from (x, y, z) to $(x+m, y+n, z+p)$ as described in part(a) is $\boxed{\dfrac{(m+n+p)!}{m!\cdot n!\, p!}}$

13) ⓐ Number of paths from $(0,0)$ to $(7,7)$ is

x moves $= 7$
y moves $= 7$

$\therefore$ Number of paths $= \dfrac{14!}{7!\,7!}$

ⓑ paths from $(2,7)$ to $(9,14)$ $= \dfrac{14!}{7!\,7!}$

14) ⓐ $P(n,2) = 90$

$$\frac{n!}{(n-2)!} = 90$$

$$\frac{1\cdot 2\cdot 3\cdots(n-2)(n-1)(n)}{1\cdot 2\cdot 3\cdots(n-2)} = 90$$

$$n(n-1) = 90$$
$$n^2 - n - 90 = 0$$
$$(n-10)(n+9) = 0$$
$$n = 10, -9$$

$\therefore \underline{n = 10}$ (-ve number is not possible)

(b) $P(n,3) = 3P(n,2)$

$$\frac{n!}{(n-3)!} = 3\,\frac{n!}{(n-2)!}$$

$$\frac{(n-2)!}{(n-3)!} = 3$$

$$\frac{1\cdot 2\cdots(n-3)(n-2)}{1\cdot 2\cdots(n-3)} = 3$$

$$n-2 = 3$$

$$\underline{\underline{n = 5}}$$

(c) $2P(n,2) + 50 = P(2n,2)$

$$2\cdot\frac{n!}{(n-2)!} + 50 = \frac{(2n)!}{(2n-2)!}$$

$$\left\{\frac{2\,n(n-1)\cancel{(n-2)\cdots 1\cdot 2}}{\cancel{1\cdot 2\cdots(n-2)}}\right\} + 50 = \frac{(2n)(2n-1)\cancel{(2n-2)\cdots 1\cdot 2}}{\cancel{(2n-2)!}}$$

$$2n^2 - 2n + 50 = 4n^2 - 2n$$

$$2n^2 - 50 = 0$$

$$n^2 - 25 = 0$$

$$(n-5)(n+5) = 0$$

$$n = 5, -5$$

$$\therefore \underline{\underline{n = 5}}$$

15) (a) DATAGRAM $\rightarrow$ 8 letters in total.

A $\rightarrow$ 3 times
rest $\rightarrow$ 1 time
(DTGRM)

$\therefore$ No of ways the letters in DATAGRAM can be arranged is $P(8;3) = \underline{\underline{\frac{8!}{3!}}}$

(b) All the A's together

D(AAA)TGRM — total 6 letters.

D, T, G, R, M, (AAA) — All 1 time.

$\therefore$ Number of ways the letters can be arranged when all A's are together $= \underline{\underline{6!}}$

16 (a) SOCIOLOGICAL — total 12 letters

S – 1
O – 3
C – 2
I – 2
L – 2
G – 1
A – 1

∴ No of arrangements of SOCIOLOGICAL is $P(12; 3, 2, 2, 2) = \dfrac{12!}{(3!)(2!)(2!)(2!)}$

(b) A & G adjacent means)

$\underbrace{(AG)}_{1}^{\text{AG adjacent}}\underbrace{SOCIOLOICL}_{10}$

Total 11 letters

O – 3 times
C – 2 times
I – 2 times
L – 2 times
(AG), S, – 1 times

∴ No of arrangements

$= \dfrac{11!}{3!\,2!\,2!\,2!}$

Also AG canbe arranged (adjaent) in 2! ways

∴ The number of arrangements in SOCIOLOGICAL when A & G adjacent $= 2! \times \dfrac{11!}{3!\,2!\,2!\,2!}$

(c) $\overset{1}{\underbrace{(OOOIIA)}_{\text{vowels}}}\underbrace{SCLGCL}_{6}$

Total → 7 letters
C – 2 times
L – 2 times

∴ No of arrangements

$= \dfrac{7!}{2!\,2!}$

Also 00011 A (adjacent) can be arranged in

$$= \frac{6!}{3!\,2!} \text{ ways}$$

total - 6
0 - 3 times
1 - 2 times
A - 1 times

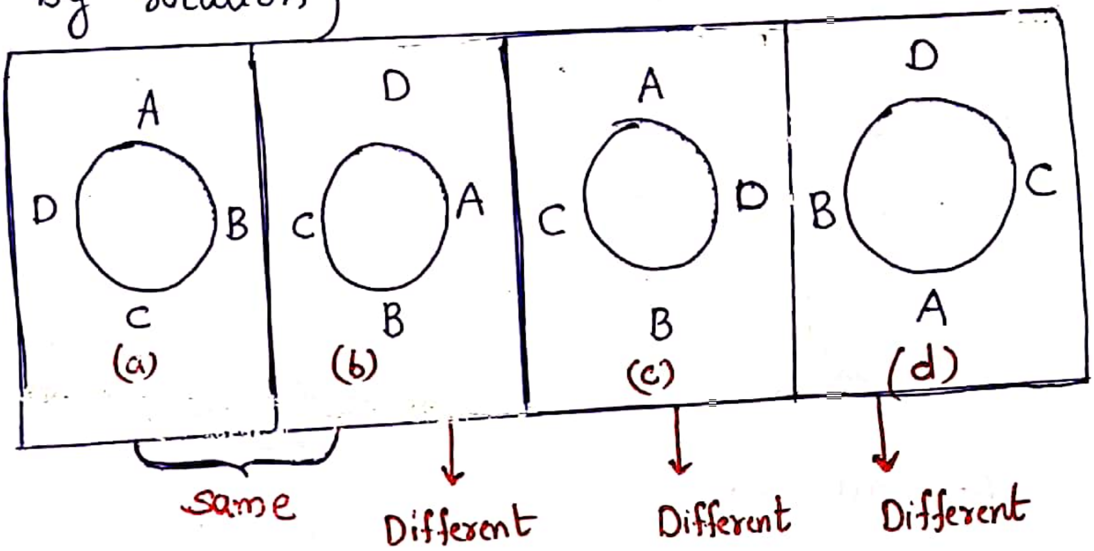
$\therefore$ Total number of arrangements of SOCIOLOGICAL when all the vowels adjacent

$$= \left(\frac{6!}{3!\,2!}\right)\left(\frac{7!}{2!\,2!}\right)$$

Circular Arrangements (Circular Permutations)

If 4 people designate as A, B, C, D are seated about a round table. In how many ways circular arrangements are possible.

(Note: The arrangements are considered the same when one can be obtained from the other by rotation)



ABCD, BCDA, CDAB, DABC, ~~etc~~ are same circular arrangement. (clockwise)

ie, each circular arrangement corresponds to 4 linear arrangements.

$\therefore$ 4 X number of circular arrangements of A, B, C, D = Number of linear arrangements of ABCD

ie 4 X circular arrangements of A, B, C, D = 4!

$\therefore$ Circular arrangements of A, B, C, D $= \frac{4!}{4} = 3!$

RESULT :-

The number of circular arrangements of n distinct objects $= \boxed{(n-1)!}$

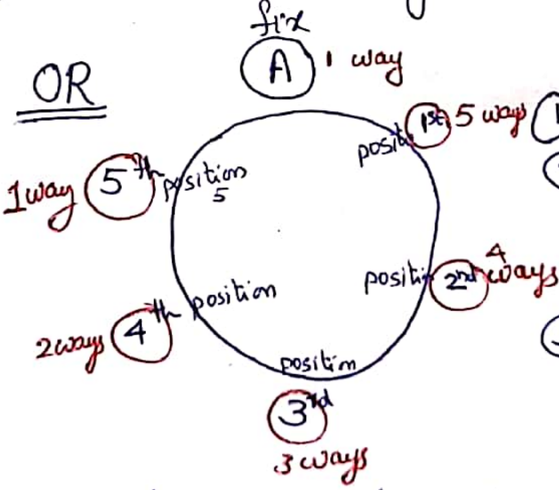
1. (a) In how many ways can six people A, B, C, D, E, F arranged about a circular table ?

(b) Suppose that the six people of the above case are 3 married couples that A, B, C are females. How many circular arrangements are possible if they sit around the table so that the sexes alternate.

Ans :- ① $n = 6$ (all are distinct)

$\therefore$ No of circular arrangements $= (n-1)! = 5! = \underline{120}$ ways — using Direct formula

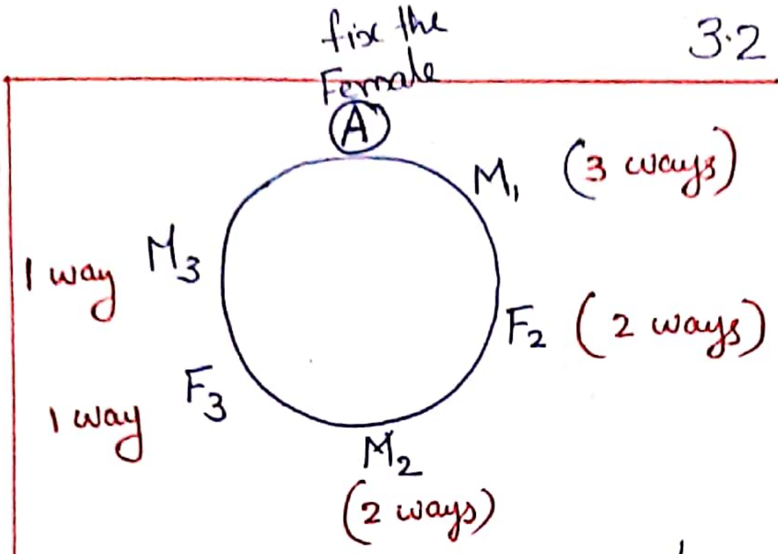
OR



① Fix A
② five locations are to be filled from (B, C, D, E, F)
③ They are the permutation of 5 distinct objects $= 5! = \underline{\underline{120}}$

By the rule of product
ie, $5 \times 4 \times 3 \times 3 \times 1 = 5! = \underline{\underline{120}}$

(b)



Alternating sexes.
OR (no two same sexes seated next to each other)

Continuing clockwise from A, by the rule of product the number of ways alternate sexes can be seated around a round table is

$$3 \times 2 \times 2 \times 1 \times 1 = \underline{12 \text{ ways}}$$

IMPORTANT

OR This idea can be interpreted as no two men or women seated next to each other.

$$= 12 \text{ ways.}$$

SAME IDEA

2(a) In how many ways can 7 people be arranged about a circular table.

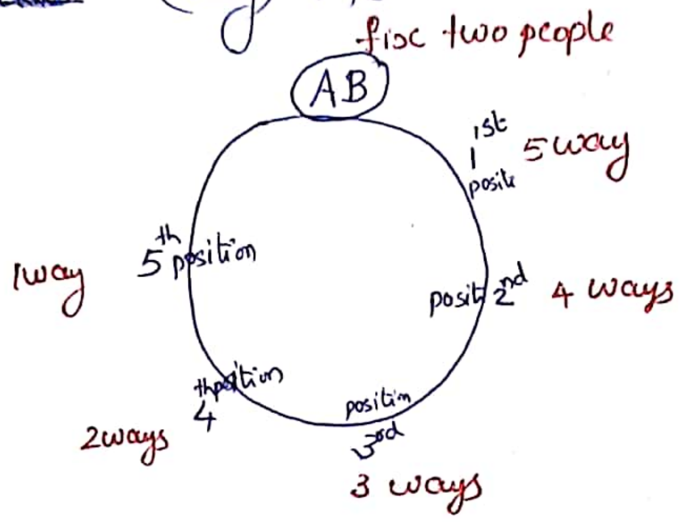
(b) If two of the people insist on sitting next to each other how many arrangements are possible

Ans:-

(a) $(n-1)! = 6! = \underline{720}$ circular arrangements

(b) A, B, C, D, E, F, G be the people.

two of them are sitting next to each other ~~[illegible]~~ (say) A, B.



$\therefore$ 5! ways.

AB can be adjacent in 2! ways.

$\therefore$ Total number of arrangements $= 2(5!) = \underline{\underline{240}}$ ways.

III COMBINATIONS

Note

When dealing with any counting problem,

1) When order is relevant } use permutations (linear arrangement) & Product Rule.

2) When order is not relevant } use combinations

Definition

The number of combinations of size 'r' from a collection of size n is

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!\,(n-r)!} \qquad 0 \le r \le n.$$

It is also denoted by $\binom{n}{r}$ Some times read as "n choose r"

Also

$C(n,0) = C(n,n) = 1$ for $n \ge 0$

$C(n,1) = C(n,(n-1)) = n$ for $n \ge 1$

$C(n,r) = \binom{n}{r} = 0$ for $0 \le n < r$

The binomial theorem:

If x and y are variables and n is a positive integer then

$$(x+y)^n = {}^nC_0\, x^0 y^n + {}^nC_1\, x^1 y^{n-1} + {}^nC_2\, x^2 y^{n-2} + \dots$$
$$+ {}^nC_{n-1}\, x^{n-1} y^1 + {}^nC_n\, x^n y^0$$
$$= \sum_{k=0}^{n} {}^nC_k\, x^k\, y^{n-k}$$

${}^nC_k \rightarrow$ binomial Coefficient

eg: If $n=4$

$$(x+y)^4 = (x+y)(x+y)(x+y)(x+y).$$

$\therefore$ the coeff x^2y^2 in the expansion of $(x+y)^4$ is the number of ways in which we can select two x's from four x's, which is ${}^4C_2 = \underline{\underline{6}}$

Note

$$(x+y)^n = \sum_{k=0}^{n} {}^nC_k\, x^k\, y^{n-k} = \sum_{k=0}^{n} {}^nC_{n-k}\, x^k\, y^{n-k}$$

1) Find the coeff x^5y^2 in the expansion of $(x+y)^7$.

Ans : ${}^7C_5 = {}^7C_2 = \underline{\underline{21}}$

2) Find the coeff. of a^5b^2 in $(2a-3b)^7$ Ans: $\underline{\underline{6048}}$

Ans: replace $2a$ by x & $-3b$ by y

ie $(x+y)^7$ $\therefore$ coeff x^5y^2 is ${}^7C_5$

$${}^7C_5\, x^5y^2 = {}^7C_5\,(2a)^5(-3b)^2 = {}^7C_5\, 2^5 \cdot (-3)^2 \cdot a^5b^2$$
$$= \underline{6048}\, a^5b^2$$

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Note:

1) $^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$

2) $^nC_0 - {}^nC_1 + {}^nC_2 - \dots + (-1)^n \, {}^nC_n = \underline{\underline{0}}$

3) Multinomial theorem:

For positive integer n, t the coefficient of $x_1^{n_1} x_2^{n_2} \dots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + \dots + x_t)^n$ is

$$\frac{n!}{n_1! \, n_2! \dots n_t!} \quad (0 \le n_i \le n, \ 1 \le i \le t, \ n_1 + n_2 + \dots + n_t = n)$$

which is called multinomial coefficient.

$$\frac{n!}{n_1! \, n_2! \dots n_t!} = \binom{n}{n_1, n_2, \dots n_t}$$

Problems

1) Coefficient of $x^2 y^2 z^3$ in $(x+y+z)^7$ is $\binom{7}{2\ 2\ 3}$

$$= \frac{7!}{2!\,2!\,3!} = 210$$

2) Coefficient of $x y z^5$ in $(x+y+z)^7$ is $\binom{7}{1\ 1\ 5}$

$$= \frac{7!}{1!\,1!\,5} = 42$$

3) coeff. of $x^3 z^4$ in $(x+y+z)^7$ is $\binom{7}{3\ 0\ 4}$

$$= \frac{7!}{3!\,0!\,4!} = \underline{\underline{35}}$$

4 coeff. $a^2 b^3 c^2 d^5$ in $(a + 2b - 3c + 2d + 5)^{16}$

$v = a \quad w = 2b \quad x = -3c \quad y = 2d \quad z = 5$

$\Rightarrow (v + w + x + y + z)^{16}$

$\therefore$ coeff. of $v^2w^3x^2y^5z^4$ is $\binom{16}{2\ 3\ 2\ 5\ 4}$

But

$$\binom{16}{2\ 3\ 2\ 5\ 4}\left(a^2(2b)^3(-3c)^2(2d)^5(5)^4\right)$$

$$=\binom{16}{2\ 3\ 2\ 5\ 4}1^2\,2^3.(-3)^2\,2^5\,5^4\,(a^2b^3c^2d^5)$$

$$= 435\,891\,456 \times 10^6\ a^2b^3c^2d^5$$

$\therefore$ Ans: $\underline{\underline{435,891\,456 \times 10^6}}$

Problem

1) Determine the coefficient of x^9y^3 in the expansion of a) $(x+y)^{12}$ b) $(x+2y)^{12}$ c) $(2x-3y)^{12}$

Ans a). coeff x^9y^3 in $(x+y)^{12} = {}^{12}C_9$

b) Coeff x^9y^3 in $(x+2y)^{12} = {}^{12}C_9 . 2^3$

c) Coeff x^9y^3 in $(2x-3y)^{12} = {}^{12}C_9 . 2^9 . (-3)^3$

2) Determine the coefficient of

a) xyz^2 in $(x+y+z)^4$ b) xyz^2 in $(w+x+y+z)^4$

c) xyz^2 in $(2x-y-z)^4$ d) xyz^{-2} in $(x-2y+3z^{-1})^4$

e) $w^3x^2yz^2$ in $(2w-x+3y-2z)^8$

Ans: a) $\binom{4}{1\ 1\ 2} = \frac{4!}{1!\,1!\,2!} = \underline{\underline{12}}$

b) $\binom{4}{0\ 1\ 1\ 2} = \underline{\underline{12}}$

c) $\binom{4}{1\ 1\ 2} \times 2 \cdot (-1)(-1)^2 = \frac{4!}{1!\,1!\,2!} \times 2 \times -1 = -24$

d) Put $a = z^{-1}$

$\therefore xyz^{-2} = xya^2$.

Coeff. xya^2 is $= \binom{4}{1\ 1\ 2}\ 1 \times (-2) \times 3^2 = \frac{4!}{1!\,1!\,2!} \times -2 \times 9 = -216$

e) $\binom{8}{3\ 2\ 1\ 2} \times 2^3 (-1)^2\ 3 \cdot (-2)^2 = \frac{8!}{3!\,2!\,1!\,2!} \times 8 \times 1 \times 3 \times 4$

$= 161280$

3) Find the Coeff. of $w^2x^2y^2z^2$ in the expansion of

HW a) $(w+x+y+z+1)^{10}$ b) $(2w-x+3y+z-2)^{12}$. and

c) $(v+w-2x+y+5z+3)^{12}$.

1. Evaluate the following

a) $C(10,4)$ b) $\binom{12}{7}$ H.W c) $C(14,12)$ H.W d) $\binom{15}{10}$ H.W

Ans:- $C(10,4) = \frac{10!}{4!\,6!} = \frac{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{(1\cdot2\cdot3\cdot4)(1\cdot2\cdot3\cdot4\cdot5\cdot6)}$

$= \underline{\underline{210}}$

b) c) d)

2) How many selections of size 2 can be made from the letters a, b, c, d, e & f

Ans:- $n = 6$
$r = 2$

$^nC_r = C(6,2) = \frac{6!}{2!\,4!} = \underline{\underline{15}}$

ab bc cd de ef
ac bd ce df
ad be cf
ae bf
af

order is not important.

3) (a) How many permutations of size 3 can one produce with the letters m, r, a, f, t

(b) List all the combinations of size 3 that result for the letters m, r, a, f, t

Ans:- a) permutations of size 3 $= P(5,3) = 5\times4\times3 = \underline{\underline{60}}$

b) Combinations of size 3 $= C(5,3) = \frac{5!}{3!\,2!} = \frac{1\cdot2\cdot3\cdot4\cdot5}{1\cdot2\cdot3 \mid 1\cdot2}$

(list them). $= \underline{\underline{10}}$

m, r, a   m, a, f   r, a, f   a, f, t
m, r, f   m, a, t   r, a, t
m, r, t   m, f, t   r, f, t

4) (a) A student taking a history exam is directed to answer any seven of 10 essay questions. In how many ways he can answer the examination.

Ans:- $n(10,7) = \frac{10!}{7!\,3!} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120$ ways.

(b) If the student must answer three questions from the first 5 and 4 questions from the last 5; In how many ways he can answer the examination.

Ans:- 3 questions from 5 = $C(5,3)$ ways
4 questions from 5 = $C(5,4)$ ways.
∴ By the Rule of product, he can complete the examinations in $\binom{5}{3}\binom{5}{4} = \frac{5!}{3!\,2!} \times \frac{5!}{4!\,1!}$
$= 10 \times 5 = 50$ ways.

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c) In how many ways a student can select 7 of the 10 questions, where at least 3 are selected from the first 5.

Ans:- $[1 \quad 2 \quad 3 \quad 4 \quad 5]$ $[6 \quad 7 \quad 8 \quad 9 \quad 10]$

First 5 — Second 5

① Select 3 qns (minimum) at least — select 4 qns = Total = 7 qns

OR

② Select 4 qns — Select 3 qns = Total - 7 qns

OR

③ Select 5 qns — select 2 qns Total - 7 qns.

∴ By the rule of sum

$$\binom{5}{3}\binom{5}{4} \overset{\text{OR}}{+} \binom{5}{4}\binom{5}{3} \overset{\text{OR}}{+} \binom{5}{5}\binom{5}{2} = \frac{5!\ \ 5!}{3!2!\ \ 4!1!} + \frac{5!\ 5!}{4!\ \ 3!2!} + \frac{5!\ \ 5!}{5!\ \ 3!2!}$$

$$= \underbrace{\sum_{i=3}^{5} \binom{5}{i}\binom{5}{7-i} = \sum_{j=2}^{4} \binom{5}{7-j}\binom{5}{j}}_{\text{In } \sum (\text{Summation (Sigma)) Notation}}$$

$= 50 + 50 + 10$

$= \underline{\underline{110}}$ selections

5) (a) A gym teacher must select nine girls from the junior and senior classes for a volleyball team. If there are 28 juniors and 25 seniors, how many ways she can select the ~~girlstes~~ girls?

H.W

(b) If two juniors and one senior are the best spikers and must be on the team. In how may ways the rest of the team can be chosen?

(c) If the team must comprise 4 juniors and 5 seniors, in how many ways she can select the team.

H.W

Ans (a) select 9 girls from 28 juniors & 25 seniors.

$\therefore$ No of ways =

(b) 2 juniors & 1 senior } (3) are already selected from a group of $28+25=53$.

$\therefore$ There remain 50 girls.

$\therefore$ We have to select 6 from 50 (rest of the team)

ie Numbers of ways $= \binom{50}{6} = \frac{50!}{6!\,44!} =$

(c)

6) Example requires Either Combination OR Permutation (Arrangement)

A gym teacher must make up 4 volleyball teams of 9 girls each from a group of 36 girls. In how many ways can she select these 4 teams? Call the teams A, B, C, and D.

Ans:- COMBINATION OR PERMUTATION.

COMBINATION

Select Team A
(9 girls from 36)
No of ways = $\binom{36}{9}$

Select Team B
(9 girls from 27)
No of ways = $\binom{27}{9}$

Select Team C
(9 girls from 18)
No of ways = $\binom{18}{9}$

Select Team D
(9 girls from 9)
No of ways = $\binom{9}{9}$

∴ By Rule of product

PERMUTATION

Consider 36 students lined up as follows
1st 2nd 3rd 36th

To select 4 teams we must distribute 9 A's (AAAAAAAAA)
9 B's (BBBBBBBBB)
9 C's (CCCCCCCCC)
& 9 D's (DDDDDDDDD)

∴ The number of arrangements of 36 ~~girls~~ (letters comprising 9 each of A, B, C & D.
ie The permutation of nondistinct objects

$$= \frac{36!}{9!\,9!\,9!\,9!}$$

4 teams can be chosen in $\binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9} = \left(\frac{36!}{9!\,27!}\right)\left(\frac{27!}{9!\,18!}\right)\left(\frac{18!}{9!\,9!}\right)\left(\frac{9!}{9!\,0!}\right) = \frac{36!}{9!\,9!\,9!\,9!}$ ways.

CONCEPTS OF BOTH PERMUTATION & COMBINATION.

7) Consider the world TALLAHASSEE.
How many of these arrangements have no adjacent A's.

Ans:- The number of arrangements of the letters in TALLA HASSEE is $\frac{11!}{3!\,2!\,2!\,2!}$

A → 3 times
L → 2 times
S → 2 times
E → 2 times
T, H → once

If we avoid 'A', the number of ways the remaining letters can be arranged in $\frac{8!}{2!\,2!\,2!} = \underline{\underline{5040}}$ (T L L H S S E E)

$\uparrow_1$ T $\uparrow_2$ L $\uparrow_3$ L $\uparrow_4$ H $\uparrow_5$ S $\uparrow_6$ S $\uparrow_7$ E $\uparrow_8$ E $\uparrow_9$ is one of the 5040 ways ~~that~~ to arrange the letters with out A.

The arrows indicate nine possible locations of the three A's.
3 of these locations can be selected in

$\binom{9}{3} = \frac{9!}{3!\,6!} = \frac{7 \cdot 8 \cdot 9}{1 \cdot 2 \cdot 3} = 84$ ways.

This is also possible for the other 5039 arrangement of T, L, L, H, S, S, E, E

∴ $5040 \times 84 = 423{,}360$ arrangements of the letters in TALLAHASSEE with no consecutive A's.

8) Suppose John draws 5 cards from a standard deck of 52 cards.

(a) In how many ways can his selection result in a hand with no clubs.

(b) In how many ways can his selection contain at least one club.

Ans:- (a) John is restricted to select 5 cards from 39 cards (Removing 13 clubs from 52)

$\therefore$ No of ways he can select 5 cards that are not clubs $= \binom{39}{5}$ ways

(b)

13 clubs.	39 others
1 club	4 others
2 clubs	3 others.
3 clubs	2 others
4 clubs	1 other
5 clubs	0 other.

$\therefore$ No of ways John can select at least one club in a hand of 5

$$= \binom{13}{1}\binom{39}{4} + \binom{13}{2}\binom{39}{3} + \binom{13}{3}\binom{39}{2} + \binom{13}{4}\binom{39}{1} + \binom{13}{5}\binom{39}{0}$$

$$= \sum_{i=1}^{5} \binom{13}{i}\binom{39}{5-i} = \underline{2,023,203} \text{ ways.}$$

$$= \frac{13!}{1!\ 12!} \frac{39!}{4!\ 35!} + \frac{13!}{2!\ 11!} \frac{39!}{}$$

9) A committee of 12 is to be selected from 10 men & 10 women. In how many ways can the selection be carried out if

(a) There are no restrictions
(b) There must be 6 men & 6 women.
(c) There must be an even number of women
(d) There must be more women than men
(e) There must be at least 8 men.

10) A student is to answer seven out of 10 questions on an examination. In how many ways can he make his selection if

(a) he must answer the first 2 qns
(b) he must answer at least 4 of the first 6 qns.

11) How many arrangements of the letters in MISSISSIPPI have no consecutive S's.

12) Determine the value of each of the following summations:

(a) $\sum_{i=1}^{6} i(-1)^i$ (b) $\sum_{j=-2}^{2} j^3 - 1$ (c) $\sum_{i=0}^{10}\left[1+(-1)^i\right]$

(d) $\sum_{K=n}^{2n} (-1)^k$ where n is an odd +ve integer.

Egs related to coding theory, computer languages etc

STRING PROBLEMS

* Consider the collection of strings of length 2 made up from the alphabet 0, 1, 2.

$$\left.\begin{matrix} 00, & 01, & 02, \\ 10, & 11, & 12, \\ 20, & 21, & 22 \end{matrix}\right\}$$ are the 9 strings of length 2.

* The number of strings of length 3.

(000), (001), (010), (100), (002), (200), (020), (110), (011), (101), (112), (211), (121), (111), (220), (202), (022), (221), (212), (122), (222), (012), (102), (021), (120), (210), (201)

There are 27 strings of length three

Note: $\therefore$ In general, if n is any positive integer then the total number of strings of length n for the alphabets 0,1 and 2 is 3^n (by the rule of product)

3	3	
1st	2nd	$= 3^2 = 9$

3	3	3	
1st	II nd	3rd	$= 3^3 = 27$

Weight of a string

Eg:- $W(12) = 1+2 = 3$

$W(101) = 1+0+1 = 2$

$W(222) = 2+2+2 = 6$

1. Consider the collection of all strings of length 10 made up from 0, 1 and 2. How many of these strings have even weight?

Ans:- A string of length 10 has even weight precisely when the number of 1's in the string is even.

No 1's (zero)

(Rule of product)

2	2	2	2	2	2	2	2	2	2
1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th

$= 2^{10}$ strings.

each location can be filled with 2 or 0

2 1's

2	2	2	2	2	2	2	2
1	2	3	4	5	6	7	8

$= 2^8$ (Two locations have been specified)

The locations for 2 1's can be selected in ${}^{10}C_2$ ways.

$\therefore$ There are ${}^{10}C_2 \times 2^8$ strings are possible.

4 1's ${}^{10}C_4\, 2^6$ strings

6 1's ${}^{10}C_6\, 2^4$ strings

8 1's ${}^{10}C_8\, 2^2$ strings

10 1's ${}^{10}C_{10}$ strings.

$\therefore$ By rule of sum, the number of strings of length 10 that have even weight is

$$2^{10} + {}^{10}C_2\, 2^8 + {}^{10}C_4\, 2^6 + {}^{10}C_6\, 2^4 + {}^{10}C_8\, 2^2 + {}^{10}C_{10}$$

H.W

2\. Consider the collection of all strings of length 10 made up from the alphabet 0, 1, 2 & 3.

a) How many of these have weight 3.

b) How many of these have weight 4.

c) How many have even weight.

3\. A gym coach must select 11 seniors to play on a football team. If he can make his selection in 12,376 ways, how many seniors are eligible to play?

H.W

2	12,376
2	6,188
2	3,094
17	1547
7	91
	13

Ans
$$^{n}C_{11} = 12,376$$

$$\frac{n!}{11!\,(n-11)!} = 12,376$$

$$\frac{n!}{(n-11)!} = (11!)\,2\times2\times2\times7\times13\times17$$

$$\frac{n(n-1)(n-2)\cdots(n-10)(n-11)\cdots3\times2\cdot1}{(n-11)!} = 11!\;2\times2\times2\times7\times13\times17$$

$$\underbrace{n(n-1)(n-2)(n-3)\cdots(n-10)}_{\text{product of 11 consecutive numbers.}} = 11\times10\times9\times8\times7\times6\times5\times4\times3\times2\times2\times2\times2\times7\times13\times17$$

$$n(n-1)(n-2)\cdots(n-10) = \underbrace{17\times16\times15\times14\times13\times12\times11\times10\times9\times8\times7}_{\text{product of 11 consecutive numbers.}}$$

$$\therefore\ \underline{\underline{n=17}}$$

The total number of eligible seniors to play $= \underline{\underline{17}}$

Answer P-55, Qn no: 2.

2) (a) How many of these have <u>weight 3</u>.

out of 10 positions we can select 1 position in ${}^{10}C_1$ ways.

1	2	3	4	5	6	7	8	9	10

Case ① One $\underbrace{3}_{\text{1 position}}$ — 9 0's — ${}^{10}C_1$

Case ② one $\underbrace{2+1}_{\text{2 positions}}$ one — 8 0's — $({}^{10}C_1)({}^{9}C_1)$

Case ③ $\underbrace{1+1+1}_{\text{3 positions - 1}}$ — 7 0's — $({}^{10}C_3)$

$$\therefore \text{No of ways} = {}^{10}C_1 + ({}^{10}C_1)({}^{9}C_1) + ({}^{10}C_3)$$

$$= 10 + 10\times 9 + \frac{10\times 9\times 8}{1\cdot 2\cdot 3} = 10 + 90 + 120 = \underline{\underline{220}}$$

(b) How many of these have <u>weight 4</u>.

Case ① $(2+1+1)$: 3 positions — 7 positions 0 — $({}^{10}C_1)({}^{9}C_2)$

Case ② $(3+1)$ two positions — 8 0's — $({}^{10}C_1)({}^{9}C_1)$

Case ③ $(2+2)$ 2 positions — 8 0's — $({}^{10}C_2)$

Case ④ $(1+1+1+1)$ 4 positions 1 — 6 0's — $({}^{10}C_4)$

$$\therefore \text{Total number of ways} = ({}^{10}C_1)({}^{9}C_2) + ({}^{10}C_1)({}^{9}C_1) + ({}^{10}C_2) + ({}^{10}C_4)$$

$$= 10\times 36 + 10\times 9 + 45 + 210$$

$$= 360 + 90 + 45 + 210 = \underline{\underline{705}}$$

(c) How many have <u>even weight</u>

To get an even weight there must be an even number of odd digits.

Select even number of positions for placing 1 OR 3 and remaining positions for placing 0 OR 2. ie For placing number at any position there are two ways.

∴ The total number of arrangements for this particular combination is

2	2	2	2	2	2	2	2	2	2
I	II	III	IV	V	VI	VII	VIII	IX	X

$= 2^{10}$ arrangements.

positions for 1 OR 3 } should be an even number including 0

positions for 1 OR 3	ways	positions 0 OR 2
0	$^{10}C_0$ ways	10 positions 0 OR 2
2	$^{10}C_2$ ways	8 positions 0 OR 2
4	$^{10}C_4$ ways	6 positions 0 OR 2
6	$^{10}C_6$ ways	4 positions 0 OR 2
8	$^{10}C_8$ ways	2 positions 0 OR 2
10	$^{10}C_{10}$ ways.	0 positions 0 OR 2.

In general we can select an even number of positions from 10 positions in $^{10}C_{2i}$ $i = 0 \ldots 5$

The number of strings of length 10 haven even weight $= {}^{10}C_2 \times 2^{10} + {}^{10}C_4\, 2^{10} + {}^{10}C_6\, 2^{10} + {}^{10}C_8\, 2^{10} + {}^{10}C_{10}\, 2^{10} + {}^{10}C_0\, 2^{10}$

$= \underline{524288}$

4. Consider the collection of all strings of length 10 made up from 0, 1, 2. How many have (a) Four 0's, three 1's and three 2's
(b) At least eight 1's
(c) Weight 4.

(a) 4 0's 3 1's 3 2's

$$\left({}^{10}C_4\right)\cdot\left({}^{6}C_3\right)\cdot\left({}^{3}C_3\right) = 4200$$

(b) At least 8 1's

Case ① 8 1's 2 positions (0 OR 2)

Case ② 9 1's 1 position (0 OR 2)

Case ③ 10 1's

$$\left({}^{10}C_8\right)\cdot 2^2 + \left({}^{10}C_9\right)\cdot 2 + \left({}^{10}C_{10}\right) = \underline{\underline{201}}$$

(c) Weight 4

Case ① 1+1+1+1 6 0's

Case ② 2+2 8 0's

Case ③ 1+1+2 7 0's

$$\left(10C4\right) + \left({}^{10}C_2\right) + \left({}^{10}C_2\right)\left({}^{8}C_1\right) = \underline{\underline{615}}$$

IV COMBINATION WITH REPETITION (WITH REPLACEMENT)

The number of combinations of 'n' distinct objects taken 'r' at a time with repetition (replacement) is ${}^{n+r-1}C_r$ ie $\boxed{\binom{n+r-1}{r}}$

Example.

1) An icecream vendor sells 3 flavors of icecreams; vanilla, chocolate & Mango. 10 kids visit the shop and each of them has one icecream. How many different purchases are possible?

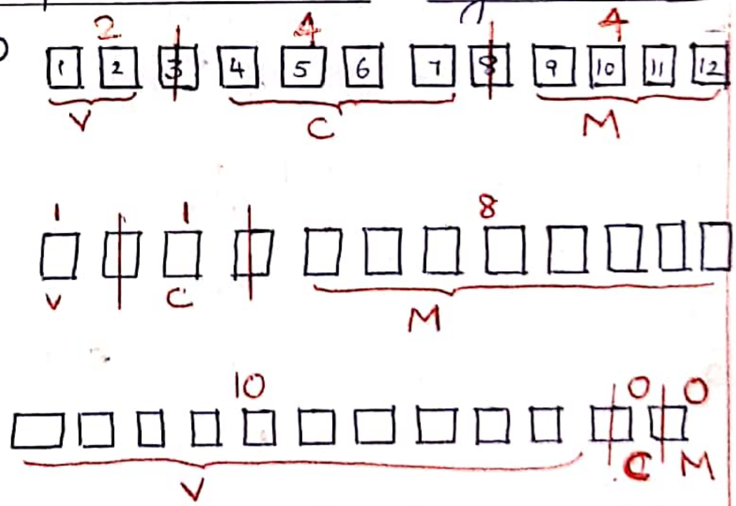
Ans:- Here $n = 3$ (vanilla, chocolate & Mango) (different flavors)
$r = 10$ (10 kids)

∴ The number of ways in which the shopkeeper can sell 10 icecreams of 3 flavors $\left.\right\}$ $= {}^{n+r-1}C_r = {}^{3+10-1}C_{10} = {}^{12}C_{10}$

$$= \frac{12!}{10!\,2!} = \frac{12 \times 11}{2 \times 1} = \underline{\underline{66}} \text{ ways.}$$

Explanation: Some of the possibilities of selling 10 icecreams.

	Vanilla	Chocolate	Mango
(i)	2	4	4
(ii)	1	1	8
(iii)	10	0	0



We are enumerating all arrangements of 12 symbols consisting of 10 □'s and 2 |'s.

∴ The number of different purchases for 10 icecreams $= \dfrac{12!}{10!\,2!}$

2. Seven highschool students stop at a restaurant where each of them has one of the following. a cheeseburger, a hot dog, a taco or a fish sandwich. How many different purchases are possible?

Ans:- $n = 4, \; r = 7$

No of pssible purchases of 7 items out of 4 items

$$= {}^{n+r-1}C_r = {}^{4+7-1}C_7 = {}^{10}C_7 = \frac{10!}{7!\,3!} = \frac{10 \times 9 \times 8}{1 \cdot 2 \cdot 3}$$

$= \underline{\underline{120}}$ different purchases.

3. A donut shop offers 20 kinds of donuts. When we entre the shop, there are at least a dozen of each kind. In how many ways one can select a dozen donuts.

Ans:- $n = 20 \quad r = 12$

Ans:- ${}^{n+r-1}C_r = {}^{20+12-1}C_{12} = \underline{\underline{{}^{31}C_{12}}}$

4. President Helen has 4 vice presidents (1) Betty (2) Goldie (3) Mary and (4) Mona. She wishes to distribute among them \$1000 in Christmas bonus checks, where each check will be written for a multiple of \$100.

In how many ways ~~she~~ can Helen distribute \$1000 among 4 vice presidents if

(a) there are no restrictions.

(b) each vice president should receive at least \$100.

ⓒ each vice president must get at least \$100 and Mona, as executive vice president gets at least \$500.

Ans:- ⓐ No restriction. (one or more VP's get nothing).

vice presidents = 4

$n = 4$,

$r = 10$ (10 units of \$100 = \$1000)

Helen is going to distribute 10 identical currencies among 4 vice presidents.

$\therefore$ No of ways = ${}^{n+r-1}C_r = {}^{4+10-1}C_{10} = {}^{13}C_{10} = \frac{13!}{10!\,3!} = \underline{\underline{286}}$

ⓑ Each VP should receive \$100 (4×100 = \$400)

Now there remain \$600 (6 units of \$100)

$n = 4$

$r = 6$

$\therefore$ No of ways = ${}^{n+r-1}C_r = {}^{4+6-1}C_6 = {}^{9}C_6 = \underline{\underline{84}}$ ways.

ⓒ Mona gets at least \$500 & others at least 100

Mona (3 others)

\$500 + \$300 = \$800 (2 units remains). $n=4, r=2$

No of ways = ${}^{n+r-1}C_r = {}^{4+2-1}C_2 = {}^{5}C_2$

$= \frac{5!}{2!\,3!} = \frac{5 \times 4}{1 \cdot 2} = \underline{\underline{10}}$ ways.

(US coin)

5) H.W In how many ways can 10 identical dimes be distributed among 5 children if

ⓐ there are no restrictions?

ⓑ each child gets at least one dime.

ⓒ the oldest child gets at least 2 dimes.

Ans:-

6) H-W Determine how many ways can 15 identical candy bars be distributed among 5 children so that the youngest gets only one or two of them.

7) H.W Determine how many ways 20 coins can be selected from four large containers filled with pennies (British coin), nickels (US coin), dimes (US coin) and quarters (US coin). (Each container is filled with only one type of coin).

8) In how many ways can we distribute 7 bananas & 6 oranges among 4 children so that each child receives one banana.

Ans:- Give each child one banana.

$\therefore$ $7-4 = 3$ bananas left.

$n=4, \; r=3.$

$\therefore$ No of ways in which the remaining 3 bananas can be distributed among 4 children $\Big\} = {}^{n+r-1}C_r = {}^{4+3-1}C_3 = {}^{6}C_3 = \frac{6\times5\times4}{1\cdot2\cdot3} = 20$ ways.

6 oranges among 4 children

$n=4, \; r=6$

No of ways in which 6 oranges can be distributed among 4 children $\Big\}$ ${}^{4+6-1}C_6 = {}^{9}C_6 = \frac{9\times8\times7}{1\cdot2\cdot3} = 84.$

$\therefore$ By the rule of product, the number of ways to distribute 7 bananas & 6 oranges under the stated conditions $= 20\times84 = \underline{1680}$ ways

9) H.W In how many ways can a teacher distribute eight chocolate donuts and seven jelly donuts among 3 student helpers if each helper wants at least one donut of each kind?

10) Determine all integer solutions to the equation $x_1+x_2+x_3+x_4=7$ where $x_i \geq 0 \ \forall\ 1 \leq i \leq 4$.

Ans:- <u>Note</u>:- The following statements are equivalent

* (a) The number of integer solutions of the equation $x_1+x_2+\cdots+x_n = r$ $x_i \geq 0,\ 1 \leq i \leq n$
* (b) The number of selections of size 'r' with repetition from a collection of size n.
* (c) The number of ways 'r' identical objects can be distributed among n distinct containers. $\boxed{{}^{n+r-1}C_r}$

n = no of different variables = 4

$r = 7$ (RHS OR SUM)

$\therefore$ No of integer solutions $= {}^{n+r-1}C_r$

$$= {}^{10}C_7 = {}^{10}C_3$$

$$= \frac{10 \times 9 \times 8}{1 \cdot 2 \cdot 3} = \underline{\underline{120 \text{ solutions}}}$$

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11. In how many ways can one distribute 10 white marbles among six distinct containers?

Ans:- This one is same as finding the number of nonnegative integer solutions to the equation $x_1 + x_2 + \cdots + x_6 = 10$

$n = 6$ (distinct containers)
$r = 10$ (10 identical white marbles)

$\therefore$ $\left.\begin{array}{l}\text{No of ways OR} \\ \text{No of nonegative} \\ \text{integer solns}\end{array}\right\} = {}^{n+r-1}C_r = {}^{15}C_{10} = {}^{15}C_5$

$= \dfrac{15 \times 14 \times 13 \times 12 \times 11}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \underline{\underline{3003}}$ ways.

12. How many nonnegative integer solutions are there to $x_1 + x_2 + \cdots + x_6 < 10$? (inequality)

Ans:- Let $x_1 + x_2 + \cdots + x_6 + x_7 = 10 \longrightarrow (1)$ where $x_1 \cdots x_6 \geq 0$ & $x_7 > 0$

$x_1 + x_2 + \cdots + x_6 + x_7 - 1 = 9 \longrightarrow (2)$

The number of solutions of Eqn (1) is same as the number of non negative integer solutions of (2)

ie $y_1 + y_2 + \cdots + y_6 + y_7 = 9$ where $x_1 = y_1$, $x_2 = y_2$, $\vdots$, $x_6 = y_6$, $x_7 - 1 = y_7$

$n = 7$
$r = 9$

No of solns $= {}^{7+9-1}C_9 = {}^{15}C_9 = \underline{\underline{5005}}$

13. Determine the number of integer solutions of $x_1+x_2+x_3+x_4 = 32$

(a) $x_i \geq 0, \ 1 \leq i \leq 4$

(b) $x_i > 0, \ 1 \leq i \leq 4$

(c) $x_1, x_2 \geq 5, \quad x_3, x_4 \geq 7$

(d) $x_i \geq 8, \ 1 \leq i \leq 4$

(e) $x_i \geq -2, \ 1 \leq i \leq 4$

(f) $x_1, x_2, x_3 > 0, \quad 0 \leq x_4 \leq 25$

Ans- (a) The number of ways, 32 identical objects can be distributed among 4 containers.

$n=4, \ r=32 \qquad$ Ans $= \ {}^{35}C_{32} = {}^{35}C_3 =$

(b) $r = 32$ items. $x_i > 0$ means $x_1, x_2, x_3, x_4 > 0$
ie at least one item in each container.

$\therefore \ 32-4 = 28$ items remain.

$x_1+x_2+x_3+x_4 = 28 \qquad n=4, \ r=28 \qquad {}^{28+4-1}C_{28} = \underline{\underline{{}^{31}C_{28}}}$

(c) $x_1, x_2 \geq 5, \quad x_3, x_4 \geq 7$
1st container & 2nd container contain at least 5 $=10$
3rd & 4th container contain at least 7 $=14$
total 24 items are already occupied

$\therefore \ 32-24 = 8 \qquad r = 8$ (remains)
$n=4$

ie $x_1+x_2+x_3+x_4 = 8$ Ans $\underline{\underline{{}^{11}C_8}}$

(d) $x_i \geq 8$ each container has minimum 8 objects.
$\therefore$ 32 objects $\quad x_1+x_2+x_3+x_4 = 0 \quad \underline{\underline{1 \text{ way}}}$.

e) $x_i \geq -2 \quad 1 \leq i \leq 4.$

Let $y_i = x_i + 2 \quad 1 \leq i \leq 4$

$$(x_1+2) + (x_2+2) + (x_3+2) + (x_4+2) = 32+8$$

$$y_1 + y_2 + y_3 + y_4 = 40$$

$n = 4,$
$r = 40$ $\quad \therefore \; {}^{4+40-1}C_{40} = \underline{\underline{{}^{43}C_{40}}}$

f) $x_1, x_2, x_3 > 0, \quad 0 < x_4 \leq 25$

Consider $x_1, x_2, x_3, x_4 > 0$
(at least one item is occupied)

$x_1 + x_2 + x_3 + x_4 = 32 - 4 = 28$

$n = 4$
$r = 28$ $\quad {}^{n+r-1}C_r = {}^{31}C_{28}$

<u>no of solutions.</u>

$x_4 \leq 25$

Consider $x_4 \geq 26$

$x_1' + x_2' + x_3' + 26 = 32$

$x_1 + x_2 + x_3 + x_4 = 32 - 29 = 3$

$n = 4$
$r = 3$ $\quad {}^{n+r-1}C_r = \underline{\underline{{}^{6}C_3}}$ no of solutions.

$$\therefore \; \underline{\underline{{}^{31}C_{28} - {}^{6}C_3}}$$

To find the total number of terms in binomial and multinomial expansions.

1) What is the total number of terms in the expansion of $(\omega+x+y+z)^{10}$?

Ans:- Each distinct term in the expansion of $(\omega+x+y+z)^{10}$ is of the form $\binom{10}{n_1\ n_2\ n_3\ n_4}\omega^{n_1}x^{n_2}y^{n_3}z^{n_4}$

where $n_1, n_2, n_3, n_4 \geq 0$ & $n_1+n_2+n_3+n_4=10$

The last eqn $n_1+n_2+n_3+n_4=10$ can be solved in ${}^{n+r-1}C_r$ ways, where $n=4$, $r=10$

The number of non -ve integer solns of $n_1+n_2+n_3+n_4=10$

ie ${}^{13}C_{10}$ ways $= 286$ ways.

$\therefore$ There are $\underline{\underline{286}}$ terms in the expansion of $(\omega+x+y+z)^{10}$.

2) What is the total number of terms in the expansion of $(x+y)^n$?

Ans- Each term is of the form $\binom{n}{k}x^k y^{n-k}$

$$\left.\begin{array}{l}\text{No of}\\ \text{terms in}\\ (x+y)^n\end{array}\right\} = {}^{2+n-1}C_n = {}^{n+1}C_n = \frac{(n+1)!}{n!\,(1)!} = \underline{\underline{(n+1)}}$$

$\because$ $n_1+n_2=n$ (no of non -ve integer solns is ${}^{2+n-1}C_n = {}^{n+1}C_n$.

$n=2$
$r=n$

V THE PIGEONHOLE PRINCIPLE

If 'm' pigeons fly to 'n' pigeonholes and $m > n$, then at least one pigeohole has two or more pigeons occupying in it.

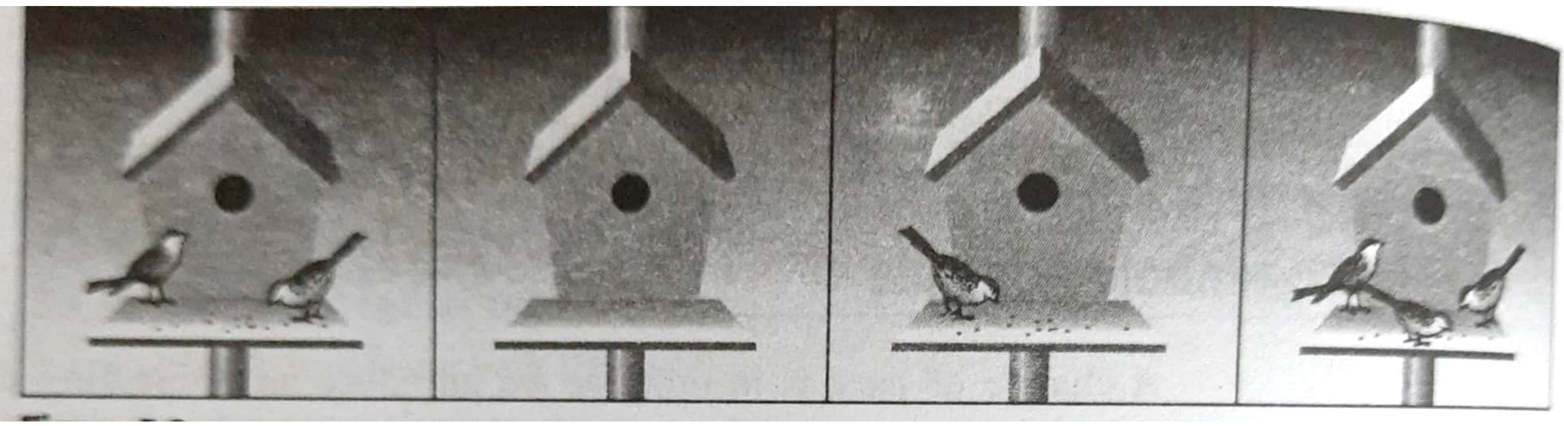
Eg:- Let number of pigeons $m = 6$ and number of pigeonholes $n = 4$. (birdhouses)

one situation is shown in the following figure. The general result actually follows by the method of proof by contradiction.

suppose the result is not true.

ie Each pigeonhole has at most one pigeon roosting in it. ie 4 pigeons are occupied out of 6. That means we have lost $6-4=2$ pigeons. ie not possible.

$\therefore$ At least one pigeon hole has 2 or more pigeons roosting in it.



1) An office employs 13 file clerks, so at least two of them must have birthdays during the same month. What plays the role of the pigeons and of the pigeonholes.

$\boxed{m>n}$

13>12

Ans:- $m = 13$ pigeons (the file clerks)
$n = 12$ pigeonholes (the months of the year)

(self survice laundry)

2) Larry returns from the laundromat with 12 pairs of socks (each pair a different colour.) in a laundry bag. Drawing the socks from the bag randomly, he will have to draw at most 13 of them to get a matched pair.

Ans:- m = pigeons (socks) = 12 pairs (24).

$\boxed{\begin{matrix} m>n \\ 24>12 \end{matrix}}$

n = pigeonholes (Colours) = 12 colours.

3) Show that if 8 people are in a room, at least 2 of them have birthdays that occur on the same day of the week.

Ans $m = 8$ people (pigeons) in the room.
$n = 7$ days (pigeon holes). in a weeks.

∴ By pigeonhole principle, if 8 pigeons fly to 7 pigeonholes, then at least ~~2 pigeonholes~~ one pigeonhole has at least 2 pigeons in it.

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4) Let $S \subset Z^+$ (Set of +ve integers) where $|S| = 37$. Then prove that S contain two elements that have the same remainder upon division by 36.

Ans:- $S = \{$A set of 37 +ve integers$\}$

When a +ve integer is divided by 36, the possible remainders are $\{0, 1, 2, 3, \ldots\ldots 35\}$ (36 remainders)

$\therefore$ Take m = pigeons = 37

n = pigeonholes = 36 Hence the result.

State pigeonhole principle also.

5) Show that any subset of size 6 from the set $S = \{1, 2, 3, \cdots 9\}$ must contain two elements whose sum is 10.

Ans:- Let m = pigeons = 6 element subset of $\{1, 2, 3 \cdots 9\}$

n = pigeonholes = $\{1,9\}, \{2,8\}, \{3,7\}, \{4,6\}, \{5\}$

= 5 subsets, which are the partitions of S

State pigeonhole principle, & hence the result.

6) Thirteen persons have first names "Rakesh, Salman, Aman" and last names "Ramana, Raju, Rao, Naidu". Show that at least 2 persons have the same first & last names.

Ans:- Total number of persons = 13 = m (pigeons)

Total number of Names = 12 ($3 \times 4 = 12$ by product rule) pigeon holes.

Hence the result by P.H principle.

Generalisation of the Pigeonhole Principle.

RESULT If m pigeons fly to n pigeonholes & $m > n$, then one of the pigeoholes must contain at least $\left\lceil \frac{m-1}{n} \right\rceil + 1$ pigeons where $\lceil x \rceil$ denotes the greatest integer less than or equal to x, which is a real number.

1. Show that if any 26 people are selected then we many choose a subset of 4 people so that all 4 were born on the same day of the week.

Ans:- $m = 26$
$n = 7$ (days in a week)

$$\therefore \left\lceil \frac{m-1}{n} \right\rceil + 1 = \left\lceil \frac{25}{7} \right\rceil + 1 = \lceil 3.57 \rceil + 1 = 3+1 = \underline{\underline{4}}$$

$\therefore$ 4 people were born on the same day of the week by generalization of pigeonhole principle.

2. Prove that in a group of six people, where any 2 people are either friends or enemies at least 3 must be mutual friends or 3 mutual strangers (Enemies).

Ans:- Let A be one of the 6 people.
Let the remaining 5 people can be accommodated in 2 rooms as FRIENDS and STRANGER to A.
Treat 5 people are pigeons & 2 rooms as pigeonholes.

$\therefore \left\lceil \frac{5-1}{2} \right\rceil + 1 = 2+1 = 3$. ie One of the room must contain at least 3 people by P.H principle

Let the room FRIENDS contains 3 people. If any 2 of them are friends, then together with A we have a set of 3 mutual friends. otherwise these 3 are mutual strangers. In either case we get the required condition.

If the room labelled STRANGERS contain 3 people we get a required conclusion by the similar argument.

3. An auditorium has a seating capacity of 800. How many seats must be occupied to guarantee that at least two people seated in the auditorium have the same first & last initials?

26	26
1st initial	Last initial

Ans - Different ~~combinations~~ choices of 1st & Last initials = $26^2 = 676$ (26 Alphabets are there)
(676 people with different initials)
Hence inorder to choose at least 2 people seated in the auditorium with same first & last initials $= 26^2 + 1 = 677$ (or any no b/w 677 & 799)
(ie if we add 1 more, then the new person cannot have an initial not counted already)

$\therefore$ At least 677 seats must be occupied

The principle of Inclusion and Exclusion

Consider a set S with $|S| = N$ and conditions c_i, $1 \le i \le t$, each of which may be satisfied by some of the elements of S. The number of elements of S that satisfy none of the conditions c_i, $1 \le i \le t$ is denoted by

$\bar{N} = N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \dots \bar{c}_t)$ where

$$\bar{N} = N - [N(c_1) + N(c_2) + \dots + N(c_t)] + [N(c_1 c_2) + N(c_1 c_3) + \dots + N(c_{t-1} c_t)] - [N(c_1 c_2 c_3) + N(c_1 c_2 c_4) + \dots + N(c_{t-2} c_{t-1} c_t)] + \dots + [(-1)^t N(c_1 c_2 c_3 \dots c_t)]$$

OR

$$\bar{N} = N - \sum_{1 \le i \le t} N(c_i) + \sum_{1 \le i < j \le t} N(c_i c_j) - \sum_{1 \le i < j < k \le t} N(c_i c_j c_k) + \dots + (-1)^t N(c_1 c_2 \dots c_t)$$

Note:
The number of elements in S that satisfy atleast one of the condition c_i, where $1 \le i \le t$ is given by $N(c_1 \text{ or } c_2 \text{ or } \dots \text{ or } c_t)$

and $N(c_1 \text{ or } c_2 \text{ or } \dots \text{ or } c_t) = N - \bar{N}$

$S_0 = N$

$S_1 = [N(c_1) + N(c_2) + \dots + N(c_t)]$

$S_2 = [N(c_1 c_2) + N(c_1 c_3) + \dots + N(c_{t-1} c_t)]$

$S_k = \sum N(c_{i_1} \cdot c_{i_2} \dots c_{i_k})$ $1 \le k \le t$.

$\therefore$ By using this $\bar{N} = S_0 - S_1 + S_2 - \dots + (-1)^t S_t$

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1) Determine the number of positive integers n where $1 \leq n \leq 100$ and n is not divisible by 2, 3 or 5

$\therefore S = \{1, 2, 3, \dots 100\}$ $N = 100$. For $n \in S$, n satisfies

a) Condition C_1 - if n is divisible by 2
C_2 - if n is divisible by 3
C_3 - if n is divisible by 5

$\therefore$ Answer is $N(\bar{C_1}\bar{C_2}\bar{C_3}) = S_0 - S_1 + S_2 - S_3$

$= N - [N(C_1) + N(C_2) + N(C_3)] + [N(C_1C_2) + N(C_1C_3) + N(C_2C_3)]$
$- N[C_1C_2C_3]$

Let $\lfloor x \rfloor$ to denote the greatest integer less than or equal to x. for any real number x.

$N(C_1) = \left\lfloor \frac{100}{2} \right\rfloor = 50$ $(2, 4, 6, \dots 100)$

$N(C_2) = \left\lfloor \frac{100}{3} \right\rfloor = 33$

$N(C_1C_3) = \left\lfloor \frac{100}{10} \right\rfloor = 10$

$N(C_3) = \left\lfloor \frac{100}{5} \right\rfloor = 20$

$N(C_2C_3) = \left\lfloor \frac{100}{15} \right\rfloor = 6$

$N(C_1C_2C_3) = \left\lfloor \frac{100}{30} \right\rfloor = 3$

$N(C_1C_2) = \left\lfloor \frac{100}{6} \right\rfloor = 16$

Answer is $= 100 - [50 + 33 + 20] + [16 + 10 + 6] - 3$

$= \underline{\underline{26}}$

These numbers are $\{1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43,$
$47, 49, 53, 59, 61, 67, 71, 73, 77, 79, 83, 89, 91, 97\}$

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2) Out of 100 students enrolled in Freshman engineering program at central university, 35 students are enrolled in Freshman Composition and 30 are enrolled in Introduction to Economics and 9 are enrolled in both FC & IE. Find the number of students who are not taking FC and not taking IE.

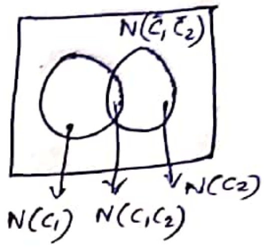
Ans: we have to find $N(\bar{C_1}\bar{C_2}) = N - [N(C_1) + N(C_2)] + N(C_1C_2)$

$N = 100$

$N(C_1) =$ students enrolled in FC $= 35$

$N(C_2) =$ students enrolled in IE $= 30$

$N(C_1C_2) = 9$ (in both FC & IE)



$N(\bar{C_1}\bar{C_2}) = 100 - [35+30] + 9 = \underline{\underline{44}}$

$$\left\{ \begin{aligned} N(\overline{C_1C_2}) &= N - N(C_1C_2) \\ &= 100 - 9 = \underline{\underline{91}} \end{aligned} \right.$$

Note:

$N(\bar{C_1}) = N - N(C_1)$

$N(\bar{C_2}) = N - N(C_2)$

$N(C_1\bar{C_2}) = N(C_1) - N(C_1C_2)$

$N(\bar{C_1}C_2) = N(C_2) - N(C_1C_2)$

$N(\bar{C_1}\bar{C_2}) = N - [N(C_1) + N(C_2)] + N(C_1C_2)$.

$N(\overline{C_1C_2}) = N - N(C_1C_2)$.

$N(\overline{C_1C_2}) \neq N(\bar{C_1}\bar{C_2})$

3 Example 2: with one more condition i.e.

C_3: student enrolled in Fundamentals of computer programming (FCP)

Also $N(C_3) = 30$, $N(C_1C_3) = 12$ $N(C_2C_3) = 10$, $N(C_1C_2C_3) = 5$

$$N(\bar{C_1}\bar{C_2}\bar{C_3}) = N - [N(C_1) + N(C_2) + N(C_3)] + [N(C_1C_2) + N(C_1C_3) + N(C_2C_3)] - N(C_1C_2C_3)$$

$= 100 - [35+30+30] + (9+11+10) - 5 = \underline{30}$

ie 30 students are not enrolled in any of the three courses (1) FC (2) IE (3) FCP

4 Example (3) with 1 condition

C_4 : student enrolled in Introduction to design (ID)

$N(C_4) = 41$ $N(C_1C_4) = 13$ $N(C_2C_4) = 14$ $N(C_3C_4) = 10$

$N(C_1C_2C_4) = 6$ $N(C_1C_3C_4) = 6$ $N(C_2C_3C_4) = 6$

$N(C_1C_2C_3C_4) = 4$

$$\therefore N(\bar{C_1}\bar{C_2}\bar{C_3}\bar{C_4}) = N - [N(C_1) + N(C_2) + N(C_3) + N(C_4)]$$
$$+ [N(C_1C_2) + N(C_1C_3) + N(C_1C_4) + N(C_2C_3) + N(C_2C_4) + N(C_3C_4)]$$
$$- [N(C_1C_2C_3) + N(C_1C_2C_4) + N(C_1C_3C_4) + N(C_2C_3C_4)]$$
$$+ N(C_1C_2C_3C_4)]$$

$$= 35 - [9+11+13] + [5+6+6] - 4 = 35 - 33 + 17 - 4 = \underline{15}$$

5) Find the number of integers between 1 and 10.000 inclusive which are divisible by 5, 6, or 8

Let A - set of all integers, A_1 = set of all integers divisible by 5

A_2 - set of all integers divisible by 6, A_3 - set of all integers divisible by 8

$\therefore |A| = 1000$ $|A_1| = \left\lfloor \frac{10000}{5} \right\rfloor = 2000$ $|A_2| = \left\lfloor \frac{10000}{6} \right\rfloor = 1666$

$|A_3| = 1250$

Note: An integer is divisible by both m & n if it is divisible by lcm (m, n)

$lcm(5,6) = 30$. $lcm(5,8) = 40$ $lcm(6,8) = 24$

$$\therefore |A_1 \cap A_2| = \left\lfloor \frac{10000}{30} \right\rfloor = 333$$

$$\therefore |A_1 \cap A_3| = \left\lfloor \frac{10000}{40} \right\rfloor = 250$$

$$\therefore |A_2 \cap A_3| = \left\lfloor \frac{10000}{24} \right\rfloor = \underline{\underline{416}}$$

$lcm(5,6,8) = 120$

$$\therefore |A_1 \cap A_2 \cap A_3| = \left\lfloor \frac{10000}{120} \right\rfloor = 83$$

$$|\bar{A_1} \cap \bar{A_2} \cap \bar{A_3}| = |A| - [|A_1| + |A_2| + |A_3|] + [|A_1 \cap A_2| + |A_2 + A_3| + |A_1 \cap A_3|] - |A_1 \cap A_2 \cap A_3|$$

$$= 10000 - (2000 + 1666 + 1250) + (333 + 250 + 416) - 83$$

$$= \underline{\underline{6000}}$$

6 In how many ways can the 6 letters of the alphabet be permuted so that none of the patterns car, dog, pun or byte occurs.

S- set of all permutation of 26 letters

$|S| = 26!$

C_1 = If the permutation contain pattern car.

C_2 = If the permutation contains pattern dog

C_3 = If the permutation contain pattern pun

C_4 = If the permutation contain pattern byte

$\underline{N(C_1)}$

Count the number of ways 24 symbols ie

car, b, d, e, ... p, q, s, t, ... x, y, z can be permuted

$\therefore N(C_1) = 24!$

Similarly $N(C_2) = N(C_3) = 24!$

$N(C_4) = 23!$ (byte, ..22 letters)

$N(C_1C_2) = 22!$ (car, dog, b, e, f, h, i, ... m, n, p, q, s, t, ... x, y, z)

$N(C_1C_3) = 22!$

$N(C_2C_3) = 22!$

$N(C_1C_4) = 21!$ $\quad N(C_2C_4) = 21!$ $\quad N(C_3C_4) = 21!$

$N(C_1C_2C_3) = 20!$ $\quad N(C_1C_3C_4) = N(C_2C_3C_4) = 20!$

$N(C_1C_2C_3C_4) = 17!$

$$N(\bar{C_1}\bar{C_2}\bar{C_3}\bar{C_4}) = 26! - [3(24!) + 23!] + [3(22!) + 3(21!)] - [20! + 3(19!)] + 17!$$

VII GENERALIZATIONS OF THE PRINCIPLE

Consider a set S with $|S| = N$ and conditions $c_1, c_2, \ldots, c_t$

Now we want to determine E_m, which denotes the number of elements in S that satisfy exactly m of the t conditions. $1 \le m \le t$

Let $E_1 = N(c_1 \bar{c}_2 \bar{c}_3 \cdots \bar{c}_t) + N(\bar{c}_1 c_2 \bar{c}_3 \cdots \bar{c}_t) + \cdots + N(\bar{c}_1 \bar{c}_2 \cdots c_t)$

$E_2 = N(c_1 c_2 \bar{c}_3 \bar{c}_4 \cdots \bar{c}_t) + N(c_1 \bar{c}_2 c_3 \bar{c}_4 \cdots \bar{c}_t) + N(\bar{c}_1 \bar{c}_2 \cdots c_{t-1} c_t)$

$\vdots$

$E_1 = N(c_1) + N(c_2) + N(c_3) - 2[N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] + 3N(c_1c_2c_3)$

$E_1 = S_1 - 2S_2 + 3S_3$	$= S_1 - \binom{2}{1}S_2 + ({}^3C_2)S_3$	When there are 3 conditions
$E_2 = S_2 - 3S_3$	$= S_2 - ({}^3C_1)S_3$	
$E_3 = S_3$		

$E_1 = S_1 - 2S_2 + 3S_3 - 4S_4$	$= S_1 - {}^2C_1 S_2 + {}^3C_2 S_3 - {}^4C_3 S_4$	When there are 4 conditions.
$E_2 = S_2 - 3S_3 + 4S_4$	$= S_2 - {}^3C_1 S_3 + {}^4C_2 S_4$	
$E_3 = S_3 - 4S_4$	$= S_3 - {}^4C_1 S_4$	
$E_4 = S_4$	$= S_4$	

$\vdots$

THEOREM: for each $1 \le m \le t$, the number of elements in S satisfy exactly m of the conditions $c_1, c_2, \ldots, c_t$ is given by $E_m = S_m - \binom{m+1}{1}S_{m+1} + \binom{m+2}{2}S_{m+2} + \cdots (-1)^{t-m}\binom{t}{t-m}S_t$

COROLLARY

If L_m denotes the number of elements of S that satisfy at least 'm' of the 't' conditions. Then we have the following formula.

$$L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} - \cdots + (-1)^{t-m} \binom{t-1}{m-1} S_t$$

1) Determine in how many ways can the letters in the word ARRANGEMENT be arranged so that
(a) there are exactly two pairs of consecutive identical letters
(b) at least 2 pairs of ~~identical~~ consecutive identical letters

Ans:- Total letters 11

A	2
R	2
E	2
N	2
G	1
T	1
M	1

Let the conditions be

c_1: two A's together
c_2: two R's together
c_3: two E's together
c_4: two N's together.

Let N = no of ways the letters in ARRANGEMENT can be permuted.

$$= \frac{11!}{2!\,2!\,2!\,2!} = 2,494,800$$

$N(C_1)$ = No of ways in which A's are together

$$= \frac{10!}{2!\,2!\,2!} = \frac{10!}{(2!)^3} = 453,600$$

$N(C_1) = N(C_2) = N(C_3) = N(C_4) = 453,600$

$N(C_1C_2)$ = No of ways in which 2 A's & 2 R's together

$$= \frac{9!}{(2!)^2} = 907,200$$

ie $N(C_iC_j) = \frac{9!}{(2!)^2}$ $\quad 1 \le i < j \le 4$

$N(C_1C_2C_3)$ = No of ways in which 2A's, 2R's, 2E's together

$N(C_iC_jC_k) = \frac{8!}{2!} = 20,160$

$N(C_1C_2C_3C_4)$ = No of ways in which 2A's, 2R's, 2E's, 2N's together

$$= 7! = 5040$$

$S_1 = {}^4C_1\,(453,600) = 1,814,400$

$S_2 = {}^4C_2\,(907,200) = 544,320$

$S_3 = {}^4C_3\,(20,160) = 80,640$

$S_4 = {}^4C_4\,5040 = 5040 =$

(a) Exactly 2 pairs of consecutive identical letters

$= E_2 = S_2 - {}^3C_1 S_3 + {}^4C_2 S_4 =$

$= 544,320 - 3(80,640) + 4(5040) = 332,640$

(b) At least 2 pairs of consecutive identical letters

$$L_2 = S_2 - ({}^{2}C_1)S_3 + ({}^{3}C_1)S_4$$

$$= 544{,}320 - 2(80{,}640) + 3(5040)$$

$$= \underline{\underline{398{,}160}}$$

HW

2) In how many ways one can arrange the letters in CORRESPONDENTS so that

(a) there is no pair of consecutive identical letters.

(b) there are exactly 2 pairs of consecutive identical letters

(c) there are at least 3 pairs of consecutive identical letters.

VIII) DERANGEMENTS: Nothing is in its right Place.

We have $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

To five places $e^{-1} = 0.36788$, & $1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!}$

$$= 0.36786.$$

ie if $K \geq 7$, $\sum_{n=0}^{K} \frac{(-1)^n}{n!}$ is a best approximation to e^{-1}

1) While at the racetrack, Ralph bets on each of the ten horses in a race to come in according to how they are favored. In how many ways can they reach the finishing line so that he loses all his bets.

Ans:- The problem is same as in how many ways we can arrange the numbers $1, 2, 3, \cdots 10$ so that 1 is not in the first place (its natural position)

2 is not in the 2^{nd} place

$\vdots$

10 is not in the 10^{th} place.

ie arrangements are called Derangements of $1, 2, 3 \cdots 10$

Here we can use the principle of exclusion & inclusion.

Let C_1: 1 is in its original position

$\vdots$

C_{10}: 10 is in its original position.

We obtain the number of derangements denoted by

$$d_{10} = N(\overline{C_1}\,\overline{C_2}\cdots\overline{C_{10}}) = \overline{N}$$

$$= N - [N(C_1) + \cdots + N(C_{10})] + [N(C_1C_2) + \cdots]$$

$$- [N(C_1C_2C_3) + \cdots]$$

$$\vdots$$

$$(-1)^{10} N(C_1C_2C_3\cdots C_{10})$$

$$= 10! - {}^{10}C_1(9!) + {}^{10}C_2(8!) - {}^{10}C_3(7!) + {}^{10}C_4(6!) - {}^{10}C_5(5!)$$
$$+ {}^{10}C_6(4!) - {}^{10}C_7(3!) + {}^{10}C_8(2!) - {}^{10}C_9(1!) + {}^{10}C_{10}(0!)$$

$$= 10! - 10(9!) + \frac{10!}{2!} - \frac{10!}{3!} + \frac{10!}{4!} - \frac{10!}{5!} + \frac{10!}{6!} - \frac{10!}{7!} \cdots]$$

$$= 10!\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \frac{1}{9!} + \frac{1}{10!}\right]$$

$$= 10!\, e^{-1} = \underline{\underline{\frac{10!}{e}}}$$

The no: of Derangements of n elements $= \dfrac{n!}{e} = D_n$

2) Find the number of derangements of 1, 2, 3, 4

$$d_4 = 4!\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right]$$

$$= 4!\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right) = 4 \times 3 \times 2 \times 1 \left(\frac{3}{8}\right) = \underline{\underline{9}}$$

$\therefore$ No of derangements $= \underline{\underline{9}}$

If we use the formula $\dfrac{n!}{e}$

$\dfrac{4!}{e} = 8.82 \simeq 9$

GENERAL FORMULA

The number of derangements of a set with n elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right] \quad \text{for } n > 1$$

For example

$$D_2 = 2!\left[1 - \frac{1}{1!} + \frac{1}{2!}\right] = 1 \quad \left(\frac{2!}{e} = 0.7357 \simeq 1\right)$$

$$D_3 = 3!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right] = 2 \quad \left(\frac{3!}{e} = 2.207 \simeq 2\right)$$

$$D_4 = 9 \quad \left(\frac{4!}{e} = 8.82 \simeq 9\right)$$

$$D_5 = 44 \quad \left(\frac{5!}{e} = 44.14 \simeq 44\right)$$

$$D_6 = 265 \quad \left(\frac{6!}{e} = 264.87 \simeq 265\right)$$

THE PROBABILITY OF DERANGEMENT OF n OBJECTS

$$= \frac{D_n}{n!} = \frac{\text{Fav. No of cases}}{\text{Total No of cases.}}$$

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3) A machine that inserts letters into envelopes goes haywire (out of control) and inserts letters randomly into envelopes. What is the probability that in a group of 100 letters

ⓐ No letter is put into correct envelope

ⓑ Exactly 1 letter is put into the correct envelope

ⓒ Exactly 98 letters are put into the correct envelope.

ⓓ Exactly 99 letters are put into the correct envelope.

ⓔ All letters are put into the correct envelopes.

Answer

(a) (No letter is put into correct envelope.)
Number of derangements of 100 elements (letters) $= D_{100}$

$\therefore$ prob. of derangements of 100 objects $= \frac{D_{100}}{100!} = \frac{\text{fav. No}}{\text{Total No}}$

(b) One letter is put into the correct envelope

$=$ Derangements of 99 objects $= D_{99}$.

D_{99} can happen in ${}^{100}C_1$ ways. $= 100$ ways

$\therefore$ derangements of 99 objects $= 100 \times D_{99}$

$\therefore$ prob of derangement $= \frac{100 \times D_{99}}{100!} = \frac{\text{fav. No}}{\text{Total No}}$.

(c) Exactly 98 letters are in the correct envelope

$=$ Derangement of ~~98~~ 2 letters ie D_2 -

D_2 can happen in ${}^{100}C_{98}$ ways or ${}^{100}C_2$

$\therefore$ Total no of $D_2 = {}^{100}C_2\, D_2$

$\therefore$ prob $= \frac{{}^{100}C_2 \times D_2}{100!}$

(d) Exactly 99 are in the correct envelope.

$\therefore$ It is not possible to misplace the remaining 1 letter

$\therefore$ All are in the correct envelope.

$\therefore$ No derangements $\therefore$ prob $= \underline{\underline{0}}$ (It is an impossible event)

(e) When all letters are in the correct envelope.

This happens only once out of 100! cases.

$\therefore$ prob $= \frac{1}{100!}$